Split, Republika Hrvatska. 15. travnja 2026.

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POVRŠINE, MOMENTI I TEŽIŠTE SREDIŠNJEG ISJEČKA ELIPSE

Osnovne definicije

$(x/a)^2 + (y/b)^2 = 1$

$\mathrm{tg}\phi = y/x$, za točku $T(x, y)$

Parametarski oblik u Cartezijevom pravokutnom k.s.:

$\boldsymbol{x = a \cdot \cos t}$

$\boldsymbol{y = b \cdot \sin t}$

$\mathrm{tg}\phi = b \cdot \sin t / [a \cdot \cos t]$

$\mathrm{tg}\phi = b/a \cdot \mathrm{tg} t$

$\mathrm{tg} t = a/b \cdot \mathrm{tg}\phi$

$t = \mathrm{arctg}(a/b \cdot \mathrm{tg}\phi) + k \cdot \pi, k \in \mathbb{Z}$

$\cos t = 1/(1+\mathrm{tg}^2 t)^{0.5} = 1/(1+a^2/b^2 \cdot \mathrm{tg}^2\phi)^{0.5}$

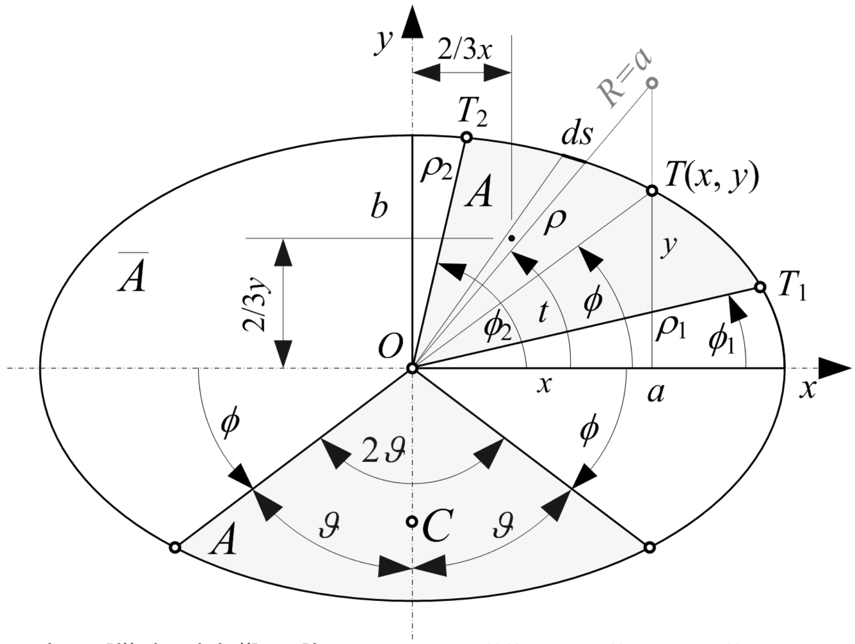
$\cos t = 1/(1+a^2\sin^2\phi/(b^2 \cdot \cos^2\phi))^{0.5}$

$\cos t = b\cos\phi/(b^2\cos^2\phi+a^2\sin^2\phi)^{0.5}$

$\sin t = \mathrm{tg} t/(1+\mathrm{tg}^2 t)^{0.5} = a/b \cdot \mathrm{tg}\phi/(1+a^2/b^2 \cdot \mathrm{tg}^2\phi)^{0.5}$

$\sin t = a\sin\phi/(b \cdot \cos\phi)/(1+a^2\sin^2\phi/(b^2 \cdot \cos^2\phi))^{0.5}$

$\sin t = a\sin\phi/(b^2\cos^2\phi+a^2\sin^2\phi)^{0.5}$



Odnos između središnjeg polarnog kuta ϕ i parametra t

$\boldsymbol{t = \mathrm{arctg}(a/b \cdot \mathrm{tg}\phi) + k \cdot \pi, k \in \mathbb{Z}}$

Površina središnjeg isječka elipse

Diferencijalni vektorski umnožak (Opći Greenov teorem)

$A = 1/2 \cdot |\int_{t_1}^{t_2}(xdy - ydx)|$

$dx = -a \cdot \sin t \cdot dt$

$dy = b \cdot \cos t \cdot dt$

$A = 1/2 \cdot |\int_{t_1}^{t_2}[a \cdot \cos t(b \cdot \cos t) \cdot dt - b \cdot \sin t(-a \cdot \sin t) \cdot dt]|$

$A = ab/2 \cdot |\int_{t_1}^{t_2}(\cos^2 t + \sin^2 t) \cdot dt|$

$A = ab/2 \cdot |\int_{t_1}^{t_2} dt|$

$\boldsymbol{A = ab|t_2 - t_1|/2}$

$\boldsymbol{\bar{A} = ab\pi - A}$

$\boldsymbol{A = ab/2 \cdot |\mathrm{arctg}(a/b \cdot \mathrm{tg}\phi_2) + k_2 \cdot \pi - \mathrm{arctg}(a/b \cdot \mathrm{tg}\phi_1) - k_1 \cdot \pi|}$

Momenti središnjeg isječka elipse

Diferencijalni vektorski umnožak (Opći Greenov teorem)

$M_x = A \cdot y_C$

$M_x = 1/3 \cdot \int_{t_1}^{t_2} y \cdot (xdy - ydx)$

$M_x = ab/3 \cdot \int_{t_1}^{t_2} b \cdot \sin t \cdot (\cos^2 t + \sin^2 t)dt$

$M_x = ab^2/3 \cdot \int_{t_1}^{t_2} \sin t \cdot dt$

$\boldsymbol{M_x = -ab^2/3 \cdot (\cos t_2 - \cos t_1)}$

$\boldsymbol{M_x = -ab^2/3 \cdot (1/(1+a^2/b^2 \cdot \mathrm{tg}^2\phi)^{0.5} - 1/(1+a^2/b^2 \cdot \mathrm{tg}^2\phi)^{0.5}}$

$M_y = A \cdot x_C$

$M_y = 1/3 \int_{t_1}^{t_2} x \cdot (xdy - ydx)$

$M_y = ab/3 \cdot \int_{t_1}^{t_2} a \cdot \cos t \cdot (\cos^2 t + \sin^2 t)dt$

$M_y = a^2b/3 \cdot \int_{t_1}^{t_2} \cos t \cdot dt$

$\boldsymbol{M_y = a^2b/3 \cdot (\sin t_2 - \sin t_1)}$

$\boldsymbol{M_y = a^2b/3[a/b \cdot \mathrm{tg}\phi_2/(1+a^2/b^2 \cdot \mathrm{tg}^2\phi_2)^{0.5} - a/b \cdot \mathrm{tg}\phi_1/(1+a^2/b^2 \cdot \mathrm{tg}^2\phi_1)^{0.5}]}$

$\boldsymbol{M_x = ab^3/3 \cdot [-\cos\phi_2/(b^2 \cdot \cos^2\phi_2+a^2 \cdot \sin^2\phi_2)^{0.5} + \cos\phi_1/(b^2 \cdot \sin^2\phi_1+a^2 \cdot \sin^2\phi_1)^{0.5}]}$

$\boldsymbol{M_y = a^3b/3 \cdot [\sin\phi_2/(b^2 \cdot \cos^2\phi_2+a^2 \cdot \sin^2\phi_2)^{0.5} - \sin\phi_1/(b^2 \cdot \sin^2\phi_1+a^2 \cdot \sin^2\phi_1)^{0.5}]}$

Težište središnjeg isječka elipse

$x_C = M_y/A$

$x_C = 1/3 \cdot a^2b(\sin t_2 - \sin t_1)/[ab|t_2 - t_1|/2]$

$\boldsymbol{x_C = 2/3 \cdot a(\sin t_2 - \sin t_1)/|t_2 - t_1|}$

$y_C = M_x/A$

$y_C = -1/3 \cdot ab^2(\cos t_2 - \cos t_1)/[ab|t_2 - t_1|/2]$

$\boldsymbol{y_C = -2/3 \cdot b(\cos t_2 - \cos t_1)/|t_2 - t_1|}$

$\boldsymbol{x_C = 2/3 \cdot a \cdot [\sin\phi_2/(b^2 \cdot \cos^2\phi_2+a^2 \cdot \sin^2\phi_2)^{0.5} - \sin\phi_1/(b^2 \cdot \sin^2\phi_1+a^2 \cdot \sin^2\phi_1)^{0.5}] / |\mathrm{arctg}(a/b \cdot \mathrm{tg}\phi_2) + k_2 \cdot \pi - \mathrm{arctg}(a/b \cdot \mathrm{tg}\phi_1) - k_1 \cdot \pi|}$

$\boldsymbol{y_C = -2/3 \cdot b \cdot [-\cos\phi_2/(b^2 \cdot \cos^2\phi_2+a^2 \cdot \sin^2\phi_2)^{0.5} + \cos\phi_1/(b^2 \cdot \sin^2\phi_1+a^2 \cdot \sin^2\phi_1)^{0.5}] / |\mathrm{arctg}(a/b \cdot \mathrm{tg}\phi_2) + k_2 \cdot \pi - \mathrm{arctg}(a/b \cdot \mathrm{tg}\phi_1) - k_1 \cdot \pi|}$

Primjeri proračuna površine središnjeg isječka elipse (od kuta $\phi_1 = 0$, $k_1 = 0$):

Do kuta $\phi_2 = \pi/2$ ($k_2 = 0$): $A = ab/2 \cdot |\mathrm{arctg}(a/b \cdot \mathrm{tg}(\pi/2)) + 0 \cdot \pi - \mathrm{arctg}(a/b \cdot \mathrm{tg}(0)) - 0 \cdot \pi| = ab \cdot |\mathrm{arctg}(\infty) + 0 - \mathrm{arctg}(0) - 0|/2 = ab \cdot |\pi/2 + 0 - 0 - 0|/2 = ab\pi/4$

Do kuta $\phi_2 = \pi$ ($k_2 = 1$): $A = ab/2 \cdot (\mathrm{arctg}(a/b \cdot \mathrm{tg}(\pi)) + 1 \cdot \pi - \mathrm{arctg}(a/b \cdot \mathrm{tg}(0)) - 0 \cdot \pi) = ab \cdot |\mathrm{arctg}(0) + \pi - \mathrm{arctg}(0) - 0|/2 = ab \cdot |0 + \pi - 0 - 0|/2 = ab\pi/2$

Do kuta $\phi_2 = 2\pi$ ($k_2 = 2$): $A = ab/2 \cdot (\mathrm{arctg}(a/b \cdot \mathrm{tg}(2\pi)) + 2 \cdot \pi - \mathrm{arctg}(a/b \cdot \mathrm{tg}(0)) - 0 \cdot \pi) = ab \cdot |\mathrm{arctg}(0) + 2\pi - \mathrm{arctg}(0) - 0|/2 = ab \cdot |0 + 2\pi - 0 - 0|/2 = ab\pi$

Simetričan, središnji isječak elipse sa središnjim kutem širine 2ϑ (za polarni kut $\phi = 3/2\pi$):

$A = ab \cdot |(3/2\pi + t) - (3/2\pi - t)|/2 = ab|3/2\pi + t - 3/2\pi + t|/2 = 2ab|t|/2 = ab[\mathrm{arctg}(a/b \cdot \mathrm{tg}\vartheta) + k \cdot \pi]$. Za $\vartheta = \pi/2$, $k = 0 \rightarrow A = ab(\mathrm{arctg}(\infty) + 0 \cdot \pi) = ab\pi/2$

$M_x = ab^3(-\cos(3/2\pi + \vartheta)/(b^2\cos^2(3/2\pi + \vartheta)+a^2\sin^2(3/2\pi + \vartheta))^{0.5} + \cos(3/2\pi - \vartheta)/(b^2\cos^2(3/2\pi - \vartheta)+a^2\sin^2(3/2\pi - \vartheta))^{0.5})/3$

$M_x = ab^3(-\sin\vartheta/(b^2\sin^2\vartheta + a^2\cos^2\vartheta)^{0.5} - \sin\vartheta/(b^2\sin^2\vartheta + a^2\cos^2\vartheta)^{0.5})/3 = -2ab^3\sin\vartheta/(b^2\sin^2\vartheta + a^2\cos^2\vartheta)^{0.5}/3$

$M_y = a^3b(\sin(3/2\pi + \vartheta)/(b^2\cos^2(3/2\pi + \vartheta)+a^2\sin^2(3/2\pi + \vartheta))^{0.5} - \sin(3/2\pi - \vartheta)/(b^2\cos^2(3/2\pi - \vartheta)+a^2\sin^2(3/2\pi - \vartheta))^{0.5})/3$

$M_y = a^3b(-\cos\vartheta/(b^2\sin^2\vartheta + a^2\cos^2\vartheta)^{0.5} + \cos\vartheta/(b^2\sin^2\vartheta + a^2\cos^2\vartheta)^{0.5})/3 = 0 \rightarrow x_T = 0$

Za $\vartheta = \pi/2 \rightarrow y_T = -2/3ab^3\sin(\pi/2)/[ab\pi/2 \cdot (b^2\sin^2\pi/2 + a^2\cos^2\pi/2)^{0.5}] = -2/3ab^3 \cdot 1/[ab\pi/2 \cdot (b^2 \cdot 1+0)^{0.5}] = -2/3ab^3/[ab\pi/2 \cdot b] = -4b/(3\pi)$