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prof. dr. sc. Dario Ban

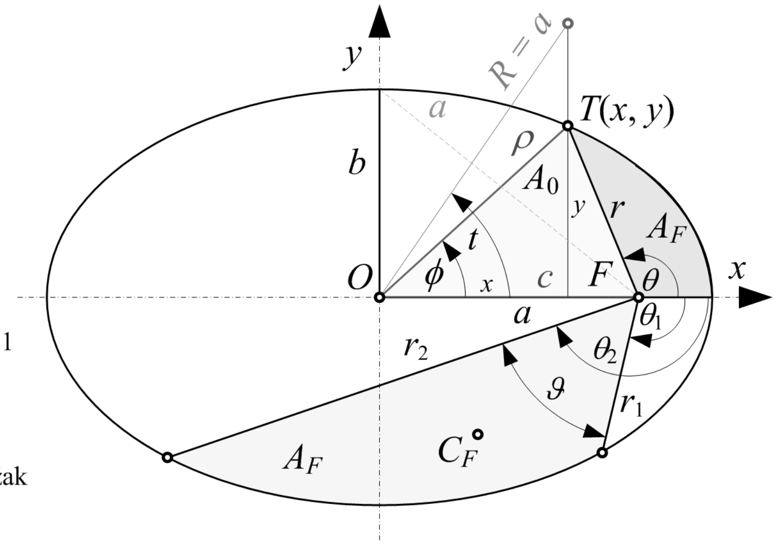
Fakultet elektrotehnike, strojarstva i brodogradnje, Sveučilište u Splitu

POVRŠINE, MOMENTI I TEŽIŠTE FOKALNOG ISJEČKA ELIPSE

Osnove

Definicija za fokus elipse

$(x/a)^2 + (y/b)^2 = 1$
$c = (a^2 - b^2)^{0.5}$
$e = c/a$ – ekscentricitet elipse
$c = e\cdot a$
$\mathrm{tg}\phi = y/x$, za točku $T(x, y)$
$x = c + r\cdot\cos\theta$
$y = r\cdot\sin\theta$
$\mathrm{tg}\phi = (c + r\cdot\cos\theta)/(r\cdot\sin\theta)$
$(c + r\cdot\cos\theta)^2/a^2 + (r\cdot\sin\theta)^2/b^2 = 1$
$r = a(1 - e^2)/(1 + e\cdot\cos\theta)$



Osnove

Definicija za ishodište k.s.

$x = a\cdot\cos t$
$y = b\cdot\sin t$
Iz jednakosti za y:
$r\cdot\sin\theta = b\cdot\sin t$
$\sin t = r/b\cdot\sin\theta$

$r = ((x - c)^2 + y^2)^{0.5}$
$r = ((a\cdot\cos t - c)^2 + b^2\cdot\sin^2 t)^{0.5}$

Odnos između fokalnog kuta θ i parametra t

$\mathbf{t = \arcsin(r/b\cdot\sin\theta)}$

Odnos između središnjeg kuta ϕ i parametra t

$t = \mathrm{arctg}(a/b\cdot\mathrm{tg}\phi)$

Površina isječka elipse

Diferencijalni vektorski umnožak
(Opći Greenov teorem)
$A_0 = 1/2\cdot|_{t1}\int^{t2}(xdy - ydx)|$
$dx = -a\cdot\sin t\cdot dt$
$dy = b\cdot\cos t\cdot dt$
$A_0 = 1/2\cdot|_{t1}\int^{t2}[a\cdot\cos t(b\cdot\cos t)\cdot dt$
$- b\cdot\sin t(-a\cdot\sin t)\cdot dt]|$
$A_0 = ab/2\cdot|_{t1}\int^{t2}(\cos^2 t + \sin^2 t)dt|$
$A_0 = ab/2\cdot|_{t1}\int^{t2}dt|$
$\mathbf{A_0 = ab|t_2 - t_1|/2}$

$\mathbf{A_0 = ab/2\cdot|\arcsin(r/b\sin\theta_2)}$
$\mathbf{- \arcsin(r/b\sin\theta_1)|}$

Površina fokalnog isječka elipse

$A_F(\theta) = A_0(\theta) - A_\Delta(\theta)$
Za 1 kut θ (slika gore):
Za $\Delta(0FT)$:
$A_\Delta(\theta) = c\cdot y/2$
$A_\Delta(\theta) = e\cdot a\cdot r\cdot\sin\theta/2$
$A_\Delta(\theta) = e\cdot a^2(1 - e^2)/(1 + e\cdot\cos\theta)\cdot\sin\theta/2$

$A_F = a\cdot b/2\cdot|\arcsin(r_2/b\sin\theta_2)$
$- \arcsin(r_1/b\sin\theta_1)| -$
$e\cdot a/2\cdot[r_2\sin\theta_2 - r_1\sin\theta_1]$

$\mathbf{A_F = ab/2\cdot|\arcsin(r_2/b\sin\theta_2)}$
$\mathbf{- \arcsin(r_1/b\sin\theta_1)| - e\cdot a^2(1 - e^2)/2}$
$\mathbf{\cdot[\sin\theta_1/(1 + e\cdot\cos\theta_1) - \sin\theta_2/(1 + e\cdot\cos\theta_2)]}$

Momenti fokalnog isječka elipse

Diferencijalni vektorski umnožak (Opći Greenov teorem)

$M_{F,x}(\theta) = M_{0,x}(\theta) - M_{\Delta,x}$
Za 1 kut θ:
$M_{0,x} = A_0\cdot y_C$
$M_{0,x} = 1/3\cdot_{t1}\int^{t2}y\cdot(xdy - ydx)$
$M_{0,x} = ab/3\cdot_{t1}\int^{t2}b\cdot\sin t\cdot(\cos^2 t + \sin^2 t)dt$
$M_{0,x} = ab^2/3\cdot_{t1}\int^{t2}\sin t\cdot dt$
Za $t_1 = 0$ i $t_2 = t \rightarrow \theta_1 = 0$ i $\theta_2 = \theta$
$\mathbf{M_{0,x} = -ab^2/3\cdot(\cos t - \cos 0) = ab^2/3\cdot(1 - \cos t)}$
$M_{0,x} = ab^2/3\cdot[1 - (1 - \sin^2 t)^{0.5}]$
$M_{0,x} = ab^2/3\cdot[1 - (1 - r^2/b^2\cdot\sin^2\theta)^{0.5}]$
$M_{0,x} = ab^2/3\cdot[b - (b^2 - r^2\cdot\sin^2\theta)^{0.5}]/b$
$M_{0,x} = ab/3\cdot[b - (b^2 - r^2\cdot\sin^2\theta)^{0.5}]$
Za $\Delta(0FT)$:
$M_{\Delta,x}(\theta) = A_\Delta\cdot x_{\Delta,C}$
$y_{\Delta,C} = 1/3\cdot\Sigma_i y_i = 1/3\cdot(0 + 0 + y(\theta)) = 1/3\cdot y(\theta)$
$M_{\Delta,x}(\theta) = c\cdot y(\theta)/2\cdot 1/3\cdot y(\theta) = c\cdot y^2(\theta)/6$
$\mathbf{M_{\Delta,x}(\theta) = c\cdot r^2\cdot\sin^2(\theta)/6}$
$M_{F,x} = -ab^2/3\cdot[(b^2 - r_2^2\sin^2\theta_2)^{0.5}$
$+ (b^2 - r_1^2\sin^2\theta_1)^{0.5}]$
$- c/6\cdot[r_2^2\sin^2(\theta_2) - r_1^2\sin^2(\theta_1)]$

$\mathbf{M_{F,x} = -ab^2/3\cdot[(b^2 - r_2^2\sin^2\theta_2)^{0.5}}$
$\mathbf{+ (b^2 - r_1^2\sin^2\theta_1)^{0.5}]}$
$\mathbf{- e\cdot a/6\cdot[r_2^2\sin^2(\theta_2) - r_1^2\sin^2(\theta_1)]}$

$M_{F,y}(\theta) = M_{0,y}(\theta) - M_{\Delta,y}(\theta)$
Za 1 kut θ:
$M_{0,y} = A_0\cdot x_C$
$M_{0,y} = 1/3_{t1}\int^{t2}x\cdot(xdy - ydx)$
$M_{0,y} = ab/3\cdot_{t1}\int^{t2}a\cdot\cos t\cdot(\cos^2 t + \sin^2 t)dt$
$M_{0,y} = a^2b/3\cdot_{t1}\int^{t2}\cos t\cdot dt$
Za $t_1 = 0$ i $t_2 = t \rightarrow \theta_1 = 0$ i $\theta_2 = \theta$
$\mathbf{M_{0,y} = a^2b/3\cdot\sin t}$
$M_{0,y} = a^2b/3\cdot r/b\cdot\sin\theta = a^2\cdot r/3\cdot\sin\theta$
Za $\Delta(0FT)$:
$M_{\Delta,y}(\theta) = A_\Delta\cdot x_{\Delta,C}$
$x_{\Delta,C} = 1/3\cdot\Sigma_i x_i = 1/3\cdot(0 + c + x(\theta))$
$x_{\Delta,C} = 1/3\cdot(c + c + r\cdot\cos(\theta))$
$M_{\Delta,y}(\theta) = c\cdot y/2\cdot 1/3\cdot(2c + r\cdot\cos(\theta))$
$\mathbf{M_{\Delta,y}(\theta) = c\cdot r\cdot\sin(\theta)\cdot(2c + r\cdot\cos(\theta))/6}$
$M_{F,y}(\theta) = a^2\cdot r\cdot\sin\theta/3$
$- c\cdot r\cdot\sin(\theta)\cdot(2c + r\cdot\cos(\theta))/6$
$M_{F,y} = a^2/3\cdot[r_2\sin\theta_2 - r_1\sin\theta_1]$
$- c/6\cdot[r_2\sin(\theta_2)\cdot(2c + r_2\cos(\theta_2))$
$- r_1\sin(\theta_1)\cdot(2c + r_1\cos(\theta_1))]$

$\mathbf{M_{F,y} = a^2/3\cdot[r_2\sin\theta_2 - r_1\sin\theta_1]}$
$\mathbf{- e\cdot a/6\cdot[r_2\sin(\theta_2)\cdot(2e\cdot a + r_2\cos(\theta_2))}$
$\mathbf{- r_1\sin(\theta_1)\cdot(2e\cdot a + r_1\cos(\theta_1))]}$

Ban

Primjeri proračuna površine isječka elipse:

- Za kut θ (od kuta 0°):

$\theta_1 = 0 \rightarrow k = 0$, $t_1 = \arcsin(\sin(0)) + 0\cdot\pi \rightarrow t_1 = 0$
$\theta_2 = \theta \rightarrow k = 0, 1, 2, \ldots \rightarrow t_2 = t = \arcsin(r/b\cdot\sin(\theta)) + k\cdot\pi$
$A = ab/2\cdot|t_2 - t_1| \rightarrow A = ab/2\cdot|t - 0| = ab/2\cdot|t|$

- Polovina elipse: Za kut π: $\phi_2 = \phi = \pi \rightarrow k = 1$

Sa slike je: $r = a + c$ i $\theta = \pi \rightarrow \sin(\theta) = 0$.
$A_0(\theta) = ab/2\cdot[\arcsin(r/b\cdot\sin(\pi)) + 1\cdot\pi] = [\arcsin(0) + \pi]$
$A_0(\theta) = ab/2\cdot\pi = ab\cdot\pi/2$
$A_F(\theta) = A_0(\theta) - A(\theta) = ab\cdot\pi/2 - c\cdot a/2\cdot 0 = ab\cdot\pi/2$

- Četvrtina elipse: Za kut $\pi/2$: $\phi_2 = \phi = \pi/2 \rightarrow k = 0$.

Sa slike je: $r = a$ i $\tan(\pi - \theta) = b/c$ ili $\sin(\pi - \theta) = b/a$, $\sin(\theta) = b/a$
$A_0(\theta) = ab/2\cdot\arcsin(a/b\cdot b/a) = ab/2\cdot\arcsin(1) = ab/2\cdot\pi/2 = ab\cdot\pi/4$
$A_F(\theta) = A_0(\theta) - A(\theta) = ab\cdot\pi/4 - c\cdot a\cdot b/a/2 = ab\cdot\pi/4 - c\cdot b/2$

- Cijela elipsa: Za kut 2π: $\phi_2 = \phi = 2\pi \rightarrow k = 2$.

Sa slike je: $r = a - c$ i $\sin(\theta) = 0$.
$A_0(\theta) = ab/2\cdot[\arcsin(r/b\cdot\sin(2\pi)) + 2\cdot\pi] = [\arcsin(0) + 2\pi]$
$A_0(\theta) = ab/2\cdot\pi = ab\cdot\pi/2$
$A_F(\theta) = A_0(\theta) - A(\theta) = ab\cdot\pi - c\cdot a/2\cdot 0 = ab\cdot\pi$